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A Fast Algorithm for Generalized Hankel Matrices Arising in Finite-Moment Problems

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ABSTRACT

Consider an $n \times n$ lower triangular matrix L whose $(i + 1)$st row is defined by the coefficients of the real polynomial $p_i(x)$ of degree i such that $\{p_i(x)\}$ is a set of orthogonal polynomials satisfying a standard three-term recurrence relation. If H is an $n \times n$ real Hankel matrix with nonsingular leading principal submatrices, then $\hat{H} = LHL^T$ will be referred to as a strongly nonsingular Hankel matrix with respect to the orthogonal polynomial basis $\{p_i(x)\}$. In this paper we develop an efficient $O(n^2)$ algorithm for the solution of a system of linear equations with a real symmetric coefficient matrix $\hat{H}$ which is a Hankel matrix with respect to a suitable orthogonal polynomial basis. This leads to an efficient method for computing polynomial approximations of an unknown function given its modified moments. © 1997 Elsevier Science Inc.

1. INTRODUCTION

In this article we present a new, efficient algorithm for the solution of a linear system with a real symmetric coefficient matrix $\hat{H}_{n-1}$ which is a strongly nonsingular Hankel matrix with respect to an orthogonal polynomial

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basis. This means that $\hat{H}_{n-1} = L_{n-1} H_{n-1} L_{n-1}^T$, where $H_{n-1} = (h_{i+j-2})$ is an $n \times n$ real strongly nonsingular Hankel matrix and $L_{n-1} = (l_{i,j})$ is a real $n \times n$ lower triangular matrix such that the polynomials $p_{i-1}(x) = (l_{i,1}, \dots, l_{i,i})(1, x, \dots, x^{i-1})^T$ satisfy a three-term recurrence

$$p_0(x) = \alpha_0, \qquad p_1(x) = \alpha_1 x - \beta_1,$$

$$p_i(x) = (\alpha_i x - \beta_i) p_{i-1}(x) - \gamma_{i-1} p_{i-2}(x),$$

with $\alpha_{i-1} > 0$ and $\gamma_i > 0$ for $i \geqslant 1$.

Hankel matrices with respect to an orthogonal polynomial basis appear naturally in the solution of some finite weighted Hausdorff moment problems (see [12], [17], and [19] for general references). Specifically, we are interested in a modified weighted finite-moment problem which consists in solving the following first-kind integral equation:

$$Mf = \mu = (\mu_1, \dots, \mu_n)^T,$$

$$(Mf)_k = \int_0^1 p_{k-1}(x) f(x) w(x)\, dx, \quad k = 1, \dots, n,$$

where the polynomials $p_{k-1}(x)$ are the orthonormal Legendre polynomials in $[0, 1]$ [1]. These polynomials satisfy the following three-term recurrence relation:

$$p_0(x) = 1, \qquad p_1(x) = 2\sqrt{3}\, x - \sqrt{3},$$

$$p_{i+1}(x) = \frac{(2i+1)\sqrt{2i+3}}{(i+1)\sqrt{2i+1}} (2x-1) p_i(x) - \frac{i\sqrt{2i+3}}{(i+1)\sqrt{2i-1}} p_{i-1}(x).$$

The vector μ is the known vector of the modified moments, $w(x)$ is a positive weight on $[0, 1]$, and $f(x)$ is an unknown function. Now, if we are looking for a polynomial approximation

$$f_k(x) = \sum_{i=0}^{k} f_i^{(k)} p_k(x), \qquad k = 0, \dots, n-1,$$

of $f(x)$, then we find that the vectors $f_k = (f_0^{(k)}, \dots, f_k^{(k)})^T$ are the solution of the following linear systems

$$\hat{H}_k f_k = (\mu_1, \dots, \mu_{k+1})^T, \qquad k = 0, \dots, n-1. \tag{1.1}$$

The matrix $\hat{H}_k$ has the form of a modified Hankel matrix, that is,

$$\hat{H}_k = L_k H_k L_k^T = (h_{i,j}), \qquad h_{i,j} = \int_0^1 p_{i-1}(x) p_{j-1}(x) w(x)\, dx,$$

where H_k is the $(k+1) \times (k+1)$ leading principal submatrix of the nth order positive definite Hankel matrix H_{n-1} with the classical moments

$$\int_0^1 x^{i+j-2} w(x)\, dx$$

of $w(x)$ along its antidiagonals; moreover, L_k is the $(k+1) \times (k+1)$ lower triangular matrix defined by the coefficients of the orthonormal Legendre polynomials $p_i(x)$ on $[0, 1]$. It is well known that L_k is the lower triangular factor of the Cholesky decomposition of the inverse of the Hilbert matrix of order $k+1$ [19].

The matrix H_k is usually ill conditioned even for small values of k [20], so that the solution of a linear system with coefficient matrix H_k should be avoided. However, in [4] D. Fasino proved that L_k is a good preconditioner for H_k in the sense that

$$\operatorname{cond}(\hat{H}_k) = \operatorname{cond}(L_k H_k L_k^T) < \frac{2k+1}{l}, \qquad \inf_{x \in [0,1]} w(x) = l > 0,$$

where cond(A) represents the spectral condition number of the matrix A, that is, the ratio between its largest and its smallest singular value. This motivates the search for numerical methods which compute a triangular factorization of the matrix $\hat{H}_{n-1}$ by avoiding any transition to the matrix H_{n-1}, or equivalently, by avoiding the transition from the modified moment problem to the classical moment problem.

The derivation of the new algorithm relies on a general approach to the solution of linear systems with structured coefficient matrices (see [8], [9], [14], and [15]). An anonymous referee pointed out that the approach of [14] had already been applied in [13] to devise a fast algorithm for Hankel matrices represented in orthogonal polynomial bases.

First, we introduce a suitable displacement operator $\mathscr{K}: R^{n\times n} \to R^{n\times n}$ having the form

$$\mathscr{K}(X) = T_n X - X T_n^T,$$

where T_n is a real tridiagonal matrix associated with the sequence $\{p_i(x)\}$. This matrix has the property that the characteristic polynomial of its left $i \times i$ principal submatrix T_i coincides with the monic polynomial associated with $p_i(x)$, that is,

$$\det(xI - T_i) = \frac{p_i(x)}{\alpha_0 \alpha_1 \cdots \alpha_i}.$$

Since $\mathscr{K}(\hat{H}_{n-1})$ is a rank-two matrix, we are then able to construct recursively the vectors ρ_i which satisfy $\hat{H}_{i-1}{}^{-1} e_i = \rho_i$, where e_i denotes the last column of the $i \times i$ identity matrix. The resulting algorithm, which will be presented in Section 2, takes $5.5n^2 + O(n)$ multiplications (divisions) and $O(n)$ storage. Moreover, it can be applied without ever explicitly forming the matrix $\hat{H}_{n-1}$, since we need only the first and the last two columns of the matrix $\hat{H}_{n-1}$ for starting with the recursions. In this way, by using the well known expansion of $p_i(x)p_j(x)$ in terms of the same $p_i(x)$ [11], the needed $O(n)$ entries of the matrix $\hat{H}_{n-1}$ can be determined at the overall cost of $O(n^2)$ arithmetic operations whenever we know the expansion of $w(x)$ in terms of the orthonormal Legendre polynomials. Finally, the solution of the equation $\hat{H}_{n-1}x = b$ can be obtained as soon as the vectors ρ_i, $i = 1, \ldots, n$, are found at the overall computational cost of $6.5n^2 + O(n)$ multiplications and $O(n)$ storage.

2. DERIVATION OF THE ALGORITHM

Let us consider a nonsingular Hankel matrix

$$H_{n-1} = \begin{pmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_1 & & \iddots & h_n \\ \vdots & \iddots & \iddots & \vdots \\ h_{n-1} & h_n & \cdots & h_{2n-2} \end{pmatrix}, \tag{2.1}$$

where we assume that $\det H_k \neq 0$ for all k, $0 \leqslant k \leqslant n-1$, i.e., H_{n-1} is a strongly nonsingular Hankel matrix.

Let $\{p_k(x)\}$ be a polynomial sequence such that $p_k(x)$ is a real polynomial of degree k which satisfies a three-term recurrence

$$p_0(x) = \alpha_0, \qquad p_1(x) = \alpha_1 x - \beta_1, \tag{2.2}$$

$$p_k(x) = (\alpha_k x - \beta_k) p_{k-1}(x) - \gamma_{k-1} p_{k-2}(x) \tag{2.3}$$

with $\alpha_{k-1} > 0$ and $\gamma_k > 0$ for $k \geqslant 1$. It is well known that $p_k(x)$ are orthogonal polynomials with respect to a suitable scalar product $\langle \cdot , \cdot \rangle$; moreover, they coincide with the appropriately normalized characteristic polynomials of the leading principal submatrices of a real tridiagonal matrix T_n, that is,

$$\det(xI - T_k) = \frac{p_k(x)}{\alpha_0 \alpha_1 \cdots \alpha_k}, \qquad k \geqslant 1,$$

where

$$T_k = \begin{pmatrix} \frac{\beta_1}{\alpha_1} & \frac{1}{\alpha_1} & & \\ \frac{\gamma_1}{\alpha_2} & \frac{\beta_2}{\alpha_2} & \ddots & \\ & \ddots & \ddots & \frac{1}{\alpha_{k-1}} \\ & & \frac{\gamma_{k-1}}{\alpha_k} & \frac{\beta_k}{\alpha_k} \end{pmatrix}.$$

The matrix T_n reduces to a symmetric one if the condition $\langle p_i(x), p_i(x)\rangle = \langle p_j(x), p_j(x)\rangle$ holds for any i and j. This condition is clearly satisfied when $p_i(x)$ are orthonormal polynomials.

Let us denote by L_k the $(k+1) \times (k+1)$ lower triangular matrix such that

$$L_k(1, x, \ldots, x^k)^T = (p_0(x), \ldots, p_k(x))^T.$$

It can be easily seen that

$$L_{n-1}^{-1} T_n L_{n-1} = F(p_n),$$

where

$$F(u) = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 \\ -\dfrac{u_0}{u_n} & \cdots & \cdots & -\dfrac{u_{n-1}}{u_n} \end{pmatrix}$$

represents the Frobenius matrix associated with the polynomial $u(x) = \sum_{i=0}^{n} u_i x^i$, $u_n \neq 0$.

In different contexts many authors [4, 6, 7, 13, 10, 18] have considered the problem of solving a linear system with a coefficient matrix which has the form of the modified Hankel matrix

$$\hat{H}_{n-1} = L_{n-1} H_{n-1} L_{n-1}^T.$$

Since H_{n-1} is nonsingular, then there exists an unique solution to the following linear system:

$$H_{n-1}\left(u_0^{(n)}, \dots, u_{n-1}^{(n)}\right)^T = -(h_n, \dots, h_{2n-1})^T,$$

where h_{2n-1} is any number. If we write

$$u_n(x) = \sum_{i=0}^{n-1} u_i^{(n)} x^i + x^n,$$

then it has been proved [16] that

$$F(u_n) H_{n-1} = H_{n-1} F(u_n)^T. \tag{2.4}$$

By replacing

$$F(u_n) = F(p_n) - e_n \delta_n^T,$$

in (2.4) we find that

$$T_n \hat{H}_{n-1} - \hat{H}_{n-1} T_n^T = L_{n-1}\left[F(p_n) H_{n-1} - H_{n-1} F(p_n)^T \right] L_{n-1}^T,$$

which implies

$$T_n \hat{H}_{n-1} - \hat{H}_{n-1} T_n^T = L_{n-1}\left(e_n \delta_n^T H_{n-1} - H_{n-1} \delta_n e_n^T \right) L_{n-1}^T.$$

In this manner we obtain that

$$T_n \hat{H}_{n-1} - \hat{H}_{n-1} T_n^T = e_n \tilde{\delta}_n^T \hat{H}_{n-1} - \hat{H}_{n-1} \tilde{\delta}_n e_n^T,$$

or, equivalently,

$$\mathscr{K}\left(\hat{H}_{n-1} \right) = e_n \hat{\delta}_n^T - \hat{\delta}_n e_n^T. \tag{2.5}$$

On considering the $k \times k$ leading principal submatrix of $\mathscr{K}(\hat{H}_{n-1})$, $1 \leqslant k \leqslant n-2$, it follows that

$$T_k \hat{H}_{k-1} - \hat{H}_{k-1} T_k^T = \frac{1}{\alpha_k}\left[\left(\hat{h}_k, \dots, \hat{h}_{2k-1} \right)^T e_k^T - e_k \left(\hat{h}_k, \dots, \hat{h}_{2k-1} \right) \right],$$

where e_k denotes the last column of the $k \times k$ identity matrix. Hence, we can write

$$\hat{H}_{k-1}^{-1} T_k - T_k^T \hat{H}_{k-1}^{-1} = \frac{1}{\alpha_k} \hat{H}_{k-1}^{-1} \left\{ r_k e_k^T - e_k r_k^T \right\} \hat{H}_{k-1}^{-1}, \tag{2.6}$$

where we set $r_k = (\hat{h}_k, \dots, \hat{h}_{2k-1})^T$.

Now, let us introduce the vectors $\rho_k = (\rho_1^{(k)}, \dots, \rho_k^{(k)})^T$, $1 \leqslant k \leqslant n$, defined by

$$\hat{H}_{k-1}^{-1} e_k = \rho_k.$$

Our recursive procedure for computing the vectors ρ_k, which define a triangular factorization of $\hat{H}_{n-1}$, relies on the following proposition.

PROPOSITION 2.1. *For $1 < k < n$ the vectors ρ_{k+1} satisfy the following recursions*:

$$\frac{1}{\alpha_k}\frac{\rho_k^{(k)}}{\rho_{k+1}^{(k+1)}}\left(\rho_1^{(k+1)},\dots,\rho_k^{(k+1)}\right)^T - \frac{1}{\alpha_{k-1}}\left(\rho_{k-1}^T,0\right)^T$$

$$= \left(\frac{1}{\alpha_{k-1}}a_k - \frac{\beta_k}{\alpha_k} + \frac{1}{\alpha_k}\frac{\rho_k^{(k+1)}}{\rho_{k+1}^{(k+1)}} + T_k^T\right)\rho_k,$$

where a_k is given by

$$e_k^T\hat{H}_{k-1}\left(\rho_{k-1}^T,0\right)^T = a_k.$$

Proof. From

$$\hat{H}_k^{\;-1}\frac{e_{k+1}}{\rho_{k+1}^{(k+1)}} = \left(\frac{\rho_1^{(k+1)}}{\rho_{k+1}^{(k+1)}},\dots,\frac{\rho_k^{(k+1)}}{\rho_{k+1}^{(k+1)}},1\right)^T,$$

we obtain that

$$\hat{H}_{k-1}^{\;-1}r_k = -\left(\frac{\rho_1^{(k+1)}}{\rho_{k+1}^{(k+1)}},\dots,\frac{\rho_k^{(k+1)}}{\rho_{k+1}^{(k+1)}}\right)^T. \tag{2.7}$$

In the view of (2.6), this implies that

$$\hat{H}_{k-1}^{\;-1}T_k e_k - T_k^T\rho^{(k)} = \frac{1}{\alpha_k}\left(\frac{\rho_k^{(k+1)}}{\rho_{k+1}^{(k+1)}}\rho_k - \frac{\rho_k^{(k)}}{\rho_{k+1}^{(k+1)}}\left(\rho_1^{(k+1)},\dots,\rho_k^{(k+1)}\right)^T\right).$$

Since we have

$$T_k e_k = \frac{\beta_k}{\alpha_k}e_k + \frac{1}{\alpha_{k-1}}\left(e_{k-1}^T,0\right)^T, \qquad k > 1,$$

we can write

$$\frac{1}{\alpha_{k-1}}\hat{H}_{k-1}^{-1}\left(e_{k-1}^T, 0\right)^T + \frac{\beta_k}{\alpha_k}\rho_k - T_k^T\rho_k$$
$$= \frac{1}{\alpha_k}\left(\frac{\rho_k^{(k+1)}}{\rho_{k+1}^{(k+1)}}\rho_k - \frac{\rho_k^{(k)}}{\rho_{k+1}^{(k+1)}}\left(\rho_1^{(k+1)}, \dots, \rho_k^{(k+1)}\right)^T\right).$$

Replacing now $\hat{H}_{k-1}^{-1}(e_{k-1}^T, 0)^T$ by

$$\chi^{(k)} = \left(\rho_{k-1}^T, 0\right)^T - a_k\rho_k, \qquad k > 1,$$

for a certain a_k, produces

$$\frac{1}{\alpha_k}\frac{\rho_k^{(k)}}{\rho_{k+1}^{(k+1)}}\left(\rho_1^{(k+1)}, \dots, \rho_k^{(k+1)}\right)^T - \frac{1}{\alpha_{k-1}}\left(\rho_{k-1}^T, 0\right)^T \tag{2.8}$$

$$= \left(\frac{1}{\alpha_{k-1}}a_k - \frac{\beta_k}{\alpha_k} + \frac{1}{\alpha_k}\frac{\rho_k^{(k+1)}}{\rho_{k+1}^{(k+1)}} + T_k^T\right)\rho_k, \qquad k > 1. \tag{2.9}$$

Comparison of $\rho_k^T\hat{H}_{k-1}\chi^{(k)}$ and $\chi^{(k)T}\hat{H}_{k-1}\rho_k$ finally reveals that a_k is given by

$$a_k = -\frac{\rho_{k-1}^{(k)}}{\rho_k^{(k)}},$$

and therefore we may determine a_k by means of

$$e_k^T\hat{H}_{k-1}\left(\rho_{k-1}^T, 0\right)^T = a_k. \qquad \blacksquare \tag{2.10}$$

In order to perform efficiently the computations (2.8)–(2.10), we may observe that the equality (2.8)–(2.9) can be rewritten in the following way:

$$\left(\frac{1}{\alpha_{k-1}}a_k - \frac{\beta_k}{\alpha_k} + \frac{1}{\alpha_k}\frac{\rho_k^{(k+1)}}{\rho_{k+1}^{(k+1)}} + T_n^T\right)\begin{pmatrix}\rho_k\\ 0\end{pmatrix}$$
$$= \frac{1}{\alpha_k}\frac{\rho_k^{(k)}}{\rho_{k+1}^{(k+1)}}\begin{pmatrix}\rho_{k+1}\\ 0\end{pmatrix} - \frac{1}{\alpha_{k-1}}\begin{pmatrix}\rho_{k-1}\\ 0\end{pmatrix}, \qquad k > 1,$$

where we extend the vector ρ_k to dimension n by adding zeros. For $k = 1$, we have

$$\left(-\frac{\beta_1}{\alpha_1} + \frac{1}{\alpha_1} a_2 + T_n^T \right) \begin{pmatrix} \rho_1 \\ 0 \end{pmatrix} = \frac{1}{\alpha_1} \frac{\rho_1^{(1)}}{\rho_2^{(2)}} \begin{pmatrix} \rho_2 \\ 0 \end{pmatrix}. \tag{2.11}$$

By means of (2.5), we find that

$$\left(\frac{1}{\alpha_{k-1}} a_k - \frac{\beta_k}{\alpha_k} + \frac{1}{\alpha_k} \frac{\rho_k^{(k+1)}}{\rho_{k+1}^{(k+1)}} - e_n \hat{\delta}_n^T + T_n \right) \hat{H}_{n-1} \begin{pmatrix} \rho_k \\ 0 \end{pmatrix}$$

$$= \frac{1}{\alpha_k} \frac{\rho_k^{(k)}}{\rho_{k+1}^{(k+1)}} \hat{H}_{n-1} \begin{pmatrix} \rho_{k+1} \\ 0 \end{pmatrix} - \frac{1}{\alpha_{k-1}} \hat{H}_{n-1} \begin{pmatrix} \rho_{k-1} \\ 0 \end{pmatrix}, \qquad k > 1,$$

and therefore we are able to compute the vectors $\hat{H}_{n-1}(\rho_k, 0)^T$ in a recursive way using only $O(n)$ storage, once we know the first and the last two columns of the matrix $\hat{H}_{n-1}$.

The resulting algorithm for determining the vectors ρ_k, $k = 1, \dots, n$, proceeds as follows:

ALGORITHM 1.

Input: the entries of the tridiagonal matrix T_n associated with the sequence of the real polynomials $p_i(x)$; the entries of the first and the last two columns of a real strongly nonsingular Hankel matrix $\hat{H}_{n-1}$ with respect to the polynomial basis $\{p_i(x)\}$.

Output: the entries of the vectors ρ_k, $k = 1, \dots, n$, which satisfy $\hat{H}_{k-1}^{-1} e_k = \rho_k$.

Determine $\hat{\delta}$ by means of (2.5) (observe that its last entry can be chosen arbitrarily). Compute $\rho_1^{(1)} = 1/\hat{h}_{1,1}$ and ρ_2 by means of (2.11).

Compute recursively for $k = 2, \dots, n-1$:

1. evaluate a_{k+1} by means of (2.10);
2. find the vector $\rho_{k+1}/\rho_{k+1}^{(k+1)}$ by means of (2.8)–(2.9);
3. determine $\rho_{k+1}^{(k+1)}$ by means of $\hat{H}_k(\rho_{k+1}/\rho_{k+1}^{(k+1)}) = e_{k+1}/\rho_{k+1}^{(k+1)}$.

The number of multiplications and divisions performed at step k of Algorithm 1 is $11k + O(1)$. The initialization phase has the cost of $O(n)$ arithmetical operations. Therefore, the number of multiplications and divisions in the whole algorithm is $5.5n^2 + O(n)$, and moreover it can be implemented using only $O(n)$ storage.

Algorithm 1 generalizes some well-known fast procedures for solving Hankel linear systems (see [5] and the references given there about Levinson-type algorithms for Hankel matrices). In fact, in the case where $\alpha_{i-1} = 1$, $\beta_i = 0$, and $\gamma_i = 0$, $i = 1, \dots, n$, then the corresponding polynomials (2.2)–(2.3) become just $p_i(x) = x^i$—which do not constitute an orthogonal set—and, moreover, T_n reduces to the shift operator $P_n = (p_{i,j})$ with $p_{i,j} = \delta_{i,j-1}$.

Now the solution of the equation $\hat{H}_{n-1}x = b$, $b = (b_1, \dots, b_n)^T$, can be obtained from the following classical result [8].

PROPOSITION 2.2. *The matrix $\hat{H}_{n-1}$ is strongly nonsingular if and only if the equations*

$$\hat{H}_{k-1}\,\rho_k = e_k, \qquad k = 1, \dots, n,$$

are solvable. In this case $e_k^T \rho_k \neq 0$, $k = 1, \dots, n$. Moreover, if the vectors z_k are defined recursively by means of

$$z_1 = b_1/\hat{h}_{11},$$

$$\eta_k = b_k - e_k^T \hat{H}_{k-1}\left(z_{k-1}^T, 0\right)^T,$$

$$z_k = \left(z_{k-1}^T, 0\right)^T + \eta_k\,\rho_k,$$

then we find that $\hat{H}_{n-1} z_n = b$.

The above recursions take $n^2 + O(n)$ multiplications, and therefore we are able to solve a linear system with coefficient matrix $\hat{H}_{n-1}$ at the overall computational cost of $6.5n^2 + O(n)$ multiplications (divisions) and $O(n)$ storage.

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